

# MS<sup>4</sup>: a BPHZ killer

N. D. Lenshina<sup>1,3</sup>, A. A. Radionov<sup>2</sup>, F. V. Tkachov<sup>2</sup>

<sup>1</sup>Lomonosov Moscow State University

<sup>2</sup>Russian Academy of Sciences

<sup>3</sup>speaker

Talk at the Bogolyubov-2019 Conference, JINR, Dubna, 11.09.2019

MS<sup>4</sup> = Minimal Subtractions in  $D = 4$

# INTRODUCTION

*a student's viewpoint*

**Reality:** BPHZ is ... **OBSCURE!**

**Ill-compatible** with calculations ( $MS_{D \neq 4}$ )

*Try to derive RG equations!*

*How  $\beta$ -function is finite in  $MS_{D \neq 4}$ ?*

On the other hand: helicity methods, supersymmetry — both  $D = 4$

## Assignment:

To study, in practical examples, a formalism – **MS<sup>4</sup>** – that was a fecund source of breakthrough results in the 80's, but remained non-mainstream (Tkachov et al.)

## Surprises:

All significant algorithms can be ported to  $MS^4$  without problems

Simpler arithmetic at  $D = 4 \rightarrow D \neq 4$  has a price!

Novel formulas for RG-functions ( $\beta, \gamma, \dots$ ) – **implications unclear...**

# OUR TWO KEY TOOLS

Two special constructions of singular generalized functions:

#1 **R-operation** for non-integrable **products** of singular functions

$$\mathbf{R} \circ G = \mathbf{r} \circ \mathbf{R}' \circ G, \quad \mathbf{R}' \circ G = \sum_{\Gamma \triangleleft G} (\theta_{\Gamma} G \setminus \Gamma) \{ \mathbf{R} \circ \Gamma \}$$

$\Gamma \triangleleft$  – senior divergent subgraphs (special **subproducts**),

(...) – smooth factor, { ... } – singular factor

Tkachov, F.V.; Vlasov, V.V.: New methods for Feynman diagrams,  
p. 151-164 in: Quarks '88: an international seminar; World Scientific, 1988

#2 **Asymptotic operation** for **products** of singular functions

$$\mathbf{As} \circ G = \mathbf{r} \circ \mathbf{As}' \circ G + c_G \delta_G, \quad \mathbf{As}' \circ G = \sum_{\Gamma \triangleleft G} (\theta_{\Gamma} \mathbf{T} \circ G \setminus \Gamma) \{ \mathbf{As} \circ \Gamma \}$$

Kuznetsov, A.N. and Tkachov, F.V.: ‘MS-scheme without dimensional regularization.’:  
Talk at ‘Quarks-88. Int. Seminar’, Tbilisi, USSR, 1988

**Intimidating?** **Not really!**

*With this toolset one can address problems ...*

# 1952 BOGOLYUBOV'S AXIOM FOR $R_{UV}$

$R_{UV}$  is an extension of functional (well-known):

$$R_{UV}T(\mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)) = T(\mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)) \\ + T(\Lambda(x,y)\mathcal{L}(z)) + T(\Lambda(x,z)\mathcal{L}(y)) + T(\Lambda(y,z)\mathcal{L}(x)) + \Lambda(x,y,z)$$

Inherently recursive ( $G \rightarrow \Gamma \rightarrow \gamma \rightarrow \dots$ )  
BUT ONLY in terms of generalized functions

RECURSION  
IS POWER

*How to actually define  $R_{UV}$ ?*

**Dilemma** (the choice depends on priorities)

**Option A (BPHZ): get rid of gen'd functions => recursion, goodbye!**

Bogolyubov, Parasyuk 1955; Hepp 1966; Zavyalov, Stepanov 1965; Zimmermann 1972  
(Hepp sectors + forest formula -> extremely cumbersome formalism, unrelated to "real life")

**Option B: embrace gen'd functions => preserve & exploit recursion**

Epstein, Glaser 1975; Tkachov, Vlasov 1986 (key tool #1)

## Option B\*: MS<sup>4</sup>

Kuznetsov, Tkachov (key tool #2, the expression for  $c_G$ )

$$R_{UV} \int dp B(p) \triangleq \lim_{\Lambda \rightarrow \infty} \int^{\Lambda} dp [B(p) - S(p)]$$

AUTO  
FINITE

$p$  – all loop momenta;  $B$  – unsubtracted integrand; “cutoff”;  $S$  shadow terms

$S$  are exactly the terms of expansion that are divergent at  $p \rightarrow \infty$

- **Recursion is hidden** in  $S$ : it contains, as coefficients, similar integrals, only simpler (graph  $\rightarrow$  subgraph). See next slide...
- One normally works with finite quantities that are **independent** of the cutoff's shape (table-like cutoff function  $\Theta(p)$ )
- Looks cumbersome  $\Rightarrow$  tricks familiar from  $MS_{D \neq 4}$  **must be revised** (unlike unthinking  $D \neq 4$ )
- But  $S$  follows  $B$  like a shadow, **rarely requires attention**. See next slide...
- And, typically, **arithmetic is rather simpler**  $\Rightarrow$   **$D \neq 4$  has a price!**  
*there are (marginal) exceptions though*

## MS<sup>4</sup>: construction of *S*hadow terms

Using the mathematical **tools** (**r**, **R**, **As**) mentioned earlier

$$B(p; \kappa) \xrightarrow{\text{asymptotic expansion at } p \rightarrow \infty} \mathbf{As}' \circ B(p; \kappa)$$

$$\mathbf{As}' \circ B(p; \kappa) \xrightarrow{\text{retain divergent terms only}} \mathbf{As}'_0 \circ B(p; \kappa)$$

Terms that generate log divergencies at  $p \rightarrow \infty$  are also log divergent at  $p \rightarrow 0$  (cf.  $\int_0^\infty dx/x$ )

$$\mathbf{As}'_0 \circ B(p; \kappa) \xrightarrow{\mathbf{r} \text{ for log divergent terms}} \mathbf{r} \circ \mathbf{As}'_0 \circ B(p; \kappa) \triangleq \mathcal{S}$$

Simplest example:

$$\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2} + m^2 \frac{1}{p^4} + \dots \rightarrow \frac{1}{p^2} + m^2 \mathbf{r} \circ \frac{1}{p^4}$$

## MS<sup>4</sup>: notes on *S* shadow terms

$$S = \mathbf{r} \circ \mathbf{A} s'_0 \circ B(p; \kappa)$$

$\mathbf{A} s'_0 \circ B(p; \kappa)$  is defined **uniquely**  $\rightarrow$  commutes with any algebra

$\mathbf{r}$  contains arbitrariness which we partially (almost fully) fix:

- Simple restrictions to meet structural requirements of pQFT
- Gauge identities, etc.
- Commutation with multiplication of  $B$  by polynomials – but one should calculate how  $\mathbf{r}$  commutes with derivatives

### Why *shadow*?

Because they follow the unsubtracted expression  $B$  like *shadows*:

$$B = \mathcal{P}_1 B_1 + \mathcal{P}_2 B_2, \quad \mathcal{P}_i - \text{any polynomials}$$

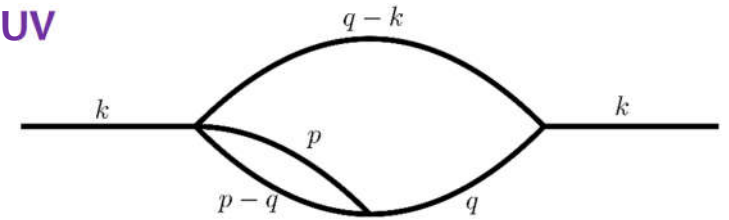
$$(B - S) = \mathcal{P}_1 (B_1 - S_1) + \mathcal{P}_2 (B_2 - S_2)$$

Kuznetsov, Tkachov 1988:

**MS<sup>4</sup> satisfies Bogolyubov's axiom for R<sub>UV</sub>**

$$\int_{\Lambda} dpdq [B(p, q; k) - \mathcal{S}] = [\text{direct calculation}]$$

$$= \int_{\Lambda} dpdq B(p, q; k) + Z_1(\Lambda) \int_{\Lambda} dq B(q; k) + Z_2'(\Lambda) + \bar{O}(\Lambda^{-1}) \bar{O}(1)$$

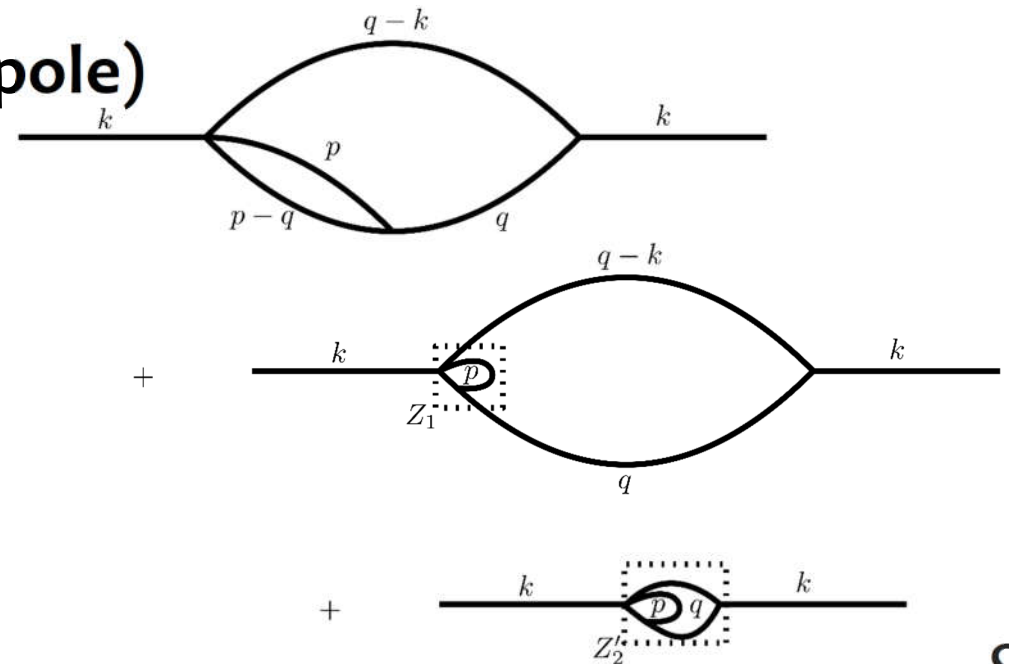


**MS<sup>4</sup> is equivalent to minimal (pole) subtractions in dimensional or analytical regularization**

$$4 \rightarrow D$$

$$\mathbf{r} \circ X \rightarrow X + \text{poles } (D - 4)^{-n}$$

$$\text{MS}^4 \rightarrow \text{MS}_{D \neq 4}$$



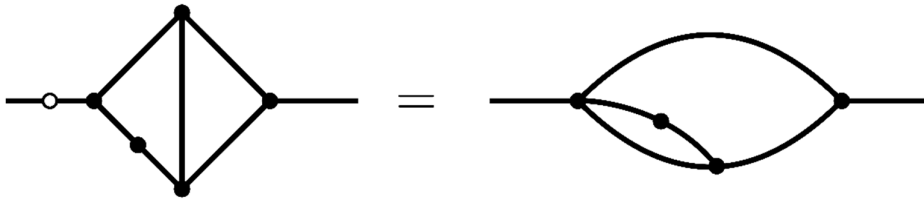
# MS<sup>4</sup>: adaptation of IBP

For instance, the classical IBP for massless self-energies (Tkachov '81)

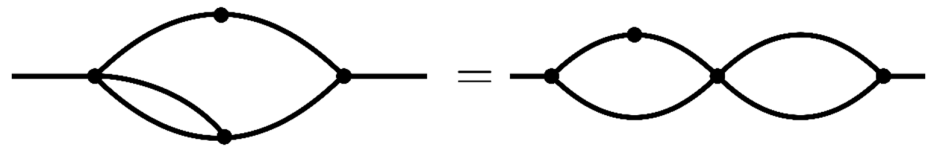
**All integrals are finite:**

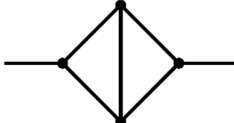
IR divergencies imply  $\mathbf{R}$ , UV divergencies imply shadow terms  $\mathcal{S}$

The IBP identities are much simpler:



And there are more of them:



However, e.g. the basic  is irreducible at  $D = 4$ .

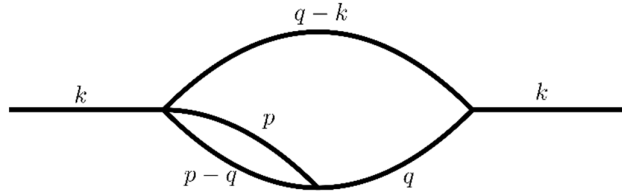
**On balance** we expect:

1. Somewhat more preparatory work, but faster computations
2. The preparatory work (irreducible integrals, etc.) is “screened” by  $D = 4$  — so one is free to use any intermediate regularization to prepare basic integrals ( $\alpha$ ,  $D \neq 4$ ,  $\ln \Lambda$ )

# MS<sup>4</sup>: How arithmetic at $D = 4$ is simpler?

First: IBP identities are obviously simpler at  $D = 4$

Second: Consider doing basic integrals — a minimal analytical regularization emerges as optimal for preparatory calculations. So, compare the 2-loop self-energy regularized in two ways:

$$\int \frac{d^4 q}{q^2 (q - k)^2} \int \frac{d^4 p}{p^{2(1+\alpha)} (p - q)^2} \sim \frac{1}{k^{2\alpha}} \left[ \frac{1}{\alpha(1-\alpha)} \right]^2$$


$$\int \frac{d^D q}{q^2 (q - k)^2} \int \frac{d^D p}{p^2 (p - q)^2} \sim \frac{1}{k^{4\varepsilon}} \frac{1}{2\varepsilon^2 (1 - 2\varepsilon)(1 - 3\varepsilon)} \frac{\Gamma^3(1 - \varepsilon) \Gamma(1 + 2\varepsilon)}{\Gamma(1 - 3\varepsilon)}$$

1. The first formula is obviously simpler  **$D \neq 4$  has a price!**
2. The disparity in complexity grows with the number of loops
3. The first formula with  $\alpha \neq 0$  occurs only in preparatory calculation of basic formulas — but **never** in IBP identities, whereas the second one **must be merged** with all the  $\varepsilon$ -dependent coefficients in IBP identities at  $D \neq 4$

## MS<sup>4</sup>: differential RG equations

$$A_{\mu}(\kappa) = \lim_{\Lambda \rightarrow \infty} \int^{\Lambda} dp \left[ B(p; \kappa) - \boxed{\mathbf{r}_{\mu}} \circ \mathbf{As}'_0 \circ B(p; \kappa) \right]$$

$\mathbf{As}'_0$  is defined uniquely – **no**  $\mu$  (!)

$\mathbf{r}_{\mu}$  – carries **all** the dependence on  $\mu$

$$\mu \frac{\partial}{\partial \mu} A_{\mu}(\kappa) = - \lim_{\Lambda \rightarrow \infty} \int^{\Lambda} dp \left[ \boxed{\mu \frac{\partial}{\partial \mu} \mathbf{r}_{\mu}} \circ \mathbf{As}'_0 \circ B(p; \kappa) \right] \quad * * * *$$

$$= (-) \int dp \boxed{\hat{\partial}} \Theta(p) \left\{ \mathbf{as}'_0 \circ B\left(p; \frac{\kappa}{\mu}\right) \right\}, \quad \boxed{\hat{\partial} \triangleq (p\partial)} \text{ is the Euler operator}$$

Using the formula for  $\mathbf{as}'_0$  one obtains (straightforwardly):

$$= (-) \sum_{\substack{\gamma = \emptyset, \dots, \Gamma \\ \gamma \neq \Gamma}} c^*_{\gamma} \left( \frac{\kappa}{\mu} \right) \left( \int dp_{\Gamma \setminus \gamma} \hat{\partial}_{\Gamma \setminus \gamma} \Theta(p_{\Gamma \setminus \gamma}) \mathbf{R}'_{\mu=1} \circ \mathbf{t}_0 \circ B_{\Gamma \setminus \gamma}(p_{\Gamma \setminus \gamma}; 0) \right)$$

## MS<sup>4</sup>: RG equation for a single multiloop integral

$$\mu \frac{\partial}{\partial \mu} A_{\mu}(\kappa) + \sum_{\substack{\gamma=\emptyset, \dots, \Gamma \\ \gamma \neq \Gamma}} c^*_{\gamma} \left( \frac{k}{\mu} \right) \times \left( \int dp_{\Gamma \setminus \gamma} \hat{\partial}_{\Gamma \setminus \gamma} \Theta(p_{\Gamma \setminus \gamma}) \mathbf{R}' \circ \mathbf{t}_0 \circ B_{\Gamma \setminus \gamma}(p_{\Gamma \setminus \gamma}; 0) \right) = 0$$

$\Gamma$  – integrand / product / “graph”

$\gamma$  – singular subproducts / “full IR subgraphs”

$\Gamma \setminus \gamma = \prod_i G_i$  – direct product of the usual UV subgraphs  $G_i$

$c^*_{\gamma}$  – integral  $\Gamma$  with all  $G_i$  contracted to points

$\left( \int dp_{\Gamma \setminus \gamma} \dots \right)$  – product of contributions from  $G_i$  to RG functions  $(\beta, \gamma, \dots)$

### Physical meaning:

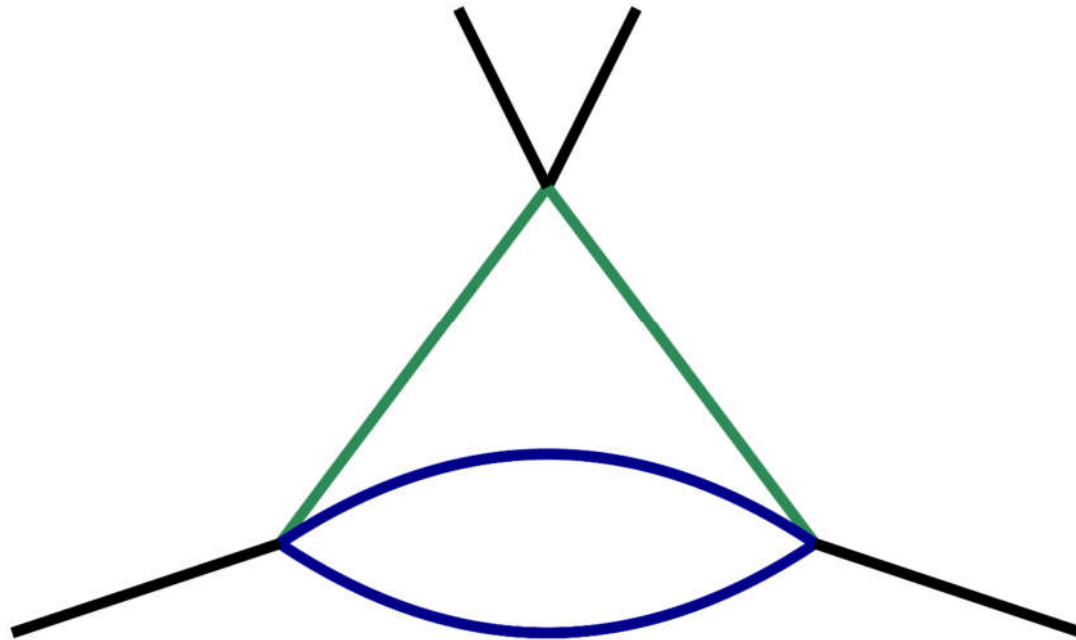
Small variation of  $\mu$  is compensated by small variations of couplings etc. — exactly as dictated by the renormalization group

Transition to complete Green functions: combinatorics following Pivovarov, Tkachov 1986

## MS<sup>4</sup>: RG equation (simple special case)

$$\mu \frac{\partial}{\partial \mu} A_{\mu}(\kappa) + \sum_{\substack{\gamma=\emptyset, \dots, \Gamma \\ \gamma \neq \Gamma}} c_{\gamma}^* \left( \frac{k}{\mu} \right) \times \int dp_{\Gamma \setminus \gamma} \hat{\partial}_{\Gamma \setminus \gamma} \Theta(p_{\Gamma \setminus \gamma}) \mathbf{R}' \circ \mathbf{t}_0 \circ B_{\Gamma \setminus \gamma}(p_{\Gamma \setminus \gamma}; 0) = 0$$

$$\mu \frac{\partial}{\partial \mu} A_{\mu}(k, k') + \int dq [B(q; k) - S_{\mu}] \times \int dp \left( p^{\alpha} \frac{\partial}{\partial p^{\alpha}} \Theta(p) \right) B(p; 0) = 0$$



## MS<sup>4</sup>: RG functions

*look at this beauty:*

$$\Delta_G = \int dp \hat{\partial} \Theta(p) \times \mathbf{R}' \circ \mathbf{t}_0 \circ B(p; 0)$$

RG functions ( $\beta, \gamma, \dots$ ) are assembled from such (finite) contributions

$G$  – UV subgraph,  $p$  – its loop momenta,  $\hat{\partial}$  – the Euler operator for  $p$ ,

$\Theta(p)$  – a smooth table-like cutoff function

$$\Theta(p)_{p \rightarrow 0} \equiv 1, \quad \Theta(p)_{p \rightarrow \infty} \equiv 0$$

$$\hat{\partial} \Theta(p)_{p \rightarrow 0} \equiv 0, \quad \hat{\partial} \Theta(p)_{p \rightarrow \infty} \equiv 0$$

*in one loop*  $\Delta$  does not depend on exact shape of  $\Theta$  (as with triangle anomaly)

$\mathbf{t}_0$  Taylor-expands in all external parameters and retains terms with log divergence at  $p = 0$

$\mathbf{R}'$  removes all divergencies except at  $p = 0$ . (key tool #1)

## MS<sup>4</sup> — SUMMARY

1. MS<sup>4</sup> is a **correct** R<sub>UV</sub> satisfying Bogolyubov's axioms
2. A full-fledged theoretical **alternative to BPHZ**
3. A direct connection to current practices (MS etc.)
4. **Minimal:** no regularization on top of inevitable cutoffs
5. **Structured:** technicalities packed into universal tools (**r, R, As**)
6. **Safe:** finite quantities most of the time
7. **Concise:** cf. derivation of RG equations (also OPE etc.)
8. Significant methods from  $D \neq 4$  allow adaptation, which, however, **requires understanding** (unlike  $D \neq 4$ )
9. Arithmetic **seems simpler** → **the fool-proof  $D \neq 4$  has a price!**
10. Some new calculational options, some flexibility
11. Unconventional view → novel formulas → **the “beauty”**  
*It would be nice to do a complete calculation... a next step.*

## SUPPLEMENTARY

$$\int dpdq [B(p)B(q) - \mathcal{S}] = \int dp [B(p) - \mathcal{S}] \times \int dq [B(q) - \mathcal{S}]$$