Numerical calculation of high-order QED contributions to the electron anomalous magnetic moment

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AMM of the electron (theory and experiment)

The measured value [2011]:

a_e=0.00115965218073(28)

The most accurate prediction (T. Kinoshita et al. [2018]):

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.001159652182032(13)(12)(720)

(α^{-1}=137.035998995(85) – independent from a_e)

Uncertainties come from:

$$A_1^{(10)},\ a_e(hadronic) + a_e(electroweak),\ \alpha$$

T. Aoyama, T. Kinoshita, M.Nio, Revised and improved value of the QED tenth-order electron anomalous magnetic moment, Physical Review D, 2018, V. 97, 036001.

My method was developed for computing $A_1^{(2n)}$

Motivation

- Independent calculation of $A_1^{(2n)}, n=5,\ldots$
- Check the validity of some hypotheses and our belief in Quantum Field Theory:
  - The contributions of gauge invariant classes are relatively small, but the contributions of individual Feynman diagrams are relatively large (in absolute value)?
  - finiteness of $A_1^{(2n)}$, behavior of the whole series etc…
  - …
- Methods of high-order calculations

Universal QED contributions

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

- J. Schwinger [1948], analytically: $A_1^{(2)} = 0.5$
- R. Karplus, N. Kroll [1949] – with a mistake

  A. Petermann [1957], C. Sommerfield [1958], analytically:

$$A_1^{(4)} = -0.328478966...$$

- ~1970…~1975, 3 loops, numerically:
  1. M. Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.

     T. Kinoshita, P. Cvitanović [1974]: $A_1^{(6)} = 1.195 \pm 0.026$
- E. Remiddi, S. Laporta et al., ~1965…1996, analytically: $A_1^{(6)} = 1.181241456...$
- T. Kinoshita et al., numerically, 2015: $A_1^{(8)} = -1.91298(84)$
- S. Laporta, semi-analytically, 2017: $A_1^{(8)} = -1.9122457649...$
- T. Kinoshita et al., numerically, 2015 (with a mistake): $A_1^{(10)} = 7.795(336)$
- T. Kinoshita et al., numerically, 2018: $A_1^{(10)} = 6.675(192)$

The method

- Subtraction procedure for removing both IR and UV divergences in Feynman-parametric space for each individual Feynman diagram
- Diagram-specific importance sampling Monte Carlo integration algorithm for diagrams without lepton loops

The subtraction procedure

- FULLY AUTOMATED AT ANY ORDER OF THE PERTURBATION SERIES.
- UV and IR divergences are eliminated point-by-point in Feynman-parametric space for each individual Feynman diagram. No regularization is required.
- Subtraction by a forest formula with linear operators. Each operator transforms Feynman amplitude of some UV-divergent subdiagram G’ (in momentum space) to the polynom with the degree that is less or equal to $\omega(G')$.
- The subtraction is equivalent to the on-shell renormalization => no residual renormalizations, no calculations of renormalization constants, no other manipulations.

Zimmermann's forest formula

- Scherbina V. [1964], Zavyalov O., Stepanov B. [1965], Zimmermann W. [1969]

$$f^{UV\text{-}free}=(1-K_1)(1-K_2)\ldots(1-K_n)f$$

K_i transforms Feynman amplitude of i-th divergent subgraph (G_i) into it's Taylor expansion up to $\omega(G_i)$ order at 0.

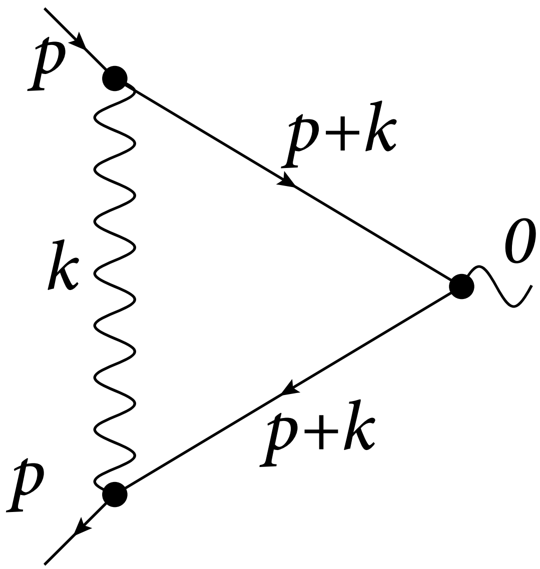
All terms with overlapping elements must be removed.

$\omega(G)$ = degree of UV divergence = $4-N_\mu-(3/2)N_e$

Disadvantages:

- IR divergences remain
- residual (physical) renormalization is required
- if we take the physical renormalization operators instead of K_j, additional IR divergences will be generated

Infrared divergences



$$p^2 = m^2$$

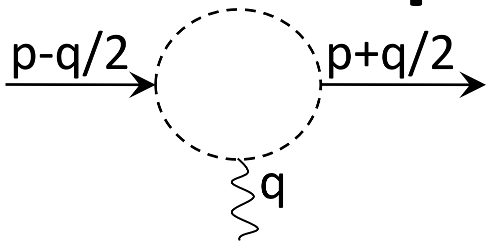
$$\Gamma_\mu = \int \frac{1}{k^2 + i\varepsilon} \gamma_\nu \frac{i(\hat{p} + \hat{k} + m)}{(p+k)^2 - m^2 + i\varepsilon} \gamma_\mu \frac{i(\hat{p} + \hat{k} + m)}{(p+k)^2 - m^2 + i\varepsilon} \gamma^\nu d^4k =$$

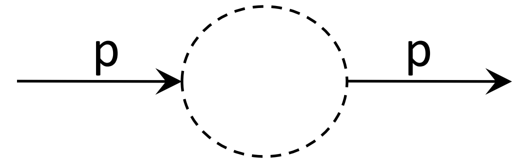
$$= a\gamma_\mu + bp_\mu + c\hat{p}p_\mu + d(\hat{p}\gamma_\mu - \gamma_\mu \hat{p})$$

all IR divergences are in *b* and *c* !!!

all UV divergences are in *a*

Operators

$$\Gamma_\mu(p,q)=$$ 

$$\Sigma(p)=$$ 

- A – projector of AMM

$$\bar{u}_2\Gamma_\mu(p,q)u_1=\bar{u}_2(f(q^2)\gamma_\mu-g(q^2)\sigma_{\mu\nu}q^\nu/(2m)+h(q^2)q_\mu)u_1$$

$$\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2,\qquad (p-q/2)^2=(p+q/2)^2=m^2$$

$$(\hat{p}-\hat{q}/2-m)u_1=(\hat{p}+\hat{q}/2-m)u_2=0$$

$$A\Gamma_\mu=\gamma_\mu\lim_{q^2\to 0}g(q^2)$$

- U – intermediate operator

$$\Gamma_\mu(p,0)=a(p^2)\gamma_\mu+b(p^2)p_\mu+c(p^2)\hat{p}p_\mu+d(p^2)(\hat{p}\gamma_\mu-\gamma_\mu\hat{p})\qquad \Sigma(p)=r(p^2)+s(p^2)\hat{p}$$

$$U\Gamma_\mu=\gamma_\mu a(m^2)\qquad U\Sigma=r(m^2)+s(m^2)\hat{p}$$

IR-safe! U preserves the Ward identity!

For the other types of divergent subgraphs, U=Taylor expansion at 0 up to ω order.

- L – on-shell renormalization for vertex-like subdiagrams

$$L\Gamma_\mu=\gamma_\mu(a(m^2)+b(m^2)m+c(m^2)m^2)$$

can produce additional IR divergences

Forest formula for AMM

A set of subgraphs of a diagram is called a **forest** if any two elements of this set don't overlap.

$\mathfrak{F}[G]$ – the set of all forests of UV-divergent subgraphs in G that contain G.

$\mathbb{I}[G]$ – the set of all vertex-like UV-divergent subgraphs in G that contains the vertex that is incident to the external photon line of G.

$$\tilde{f}_G = \sum_{\substack{F=\{G_1,\ldots,G_n\}\in\mathfrak{F}[G] \\ G'\in\mathbb{I}[G]\cap F}} (-1)^{n-1} K_{G_1}^{G'} \ldots K_{G_n}^{G'} f_G$$

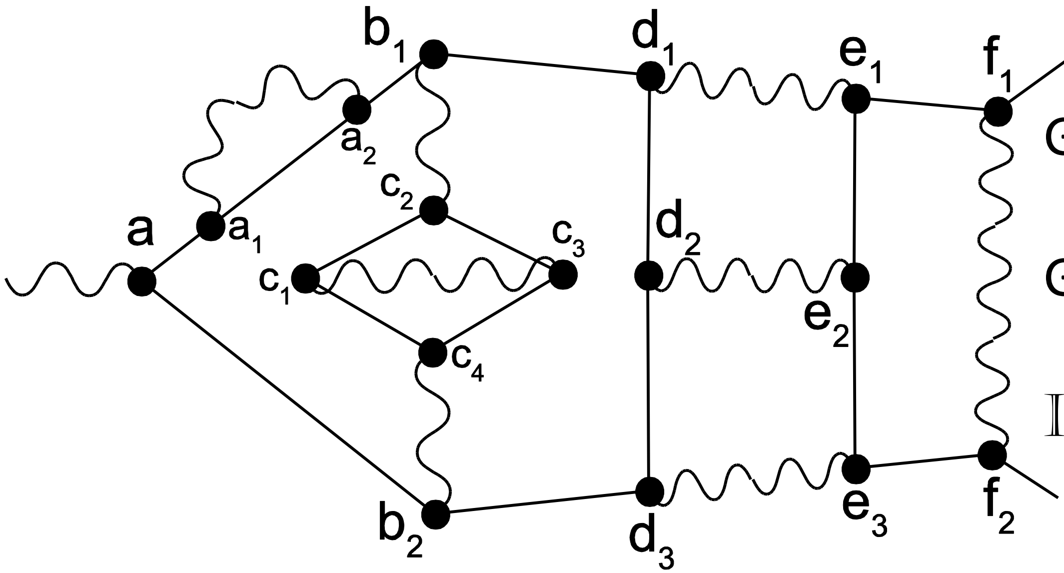
$$K_{G''}^{G'} = \begin{cases} A_{G'} \text{ for } G' = G'' \\ U_{G''} \text{ for } G'' \notin \mathbb{I}[G], \text{ or } G'' \subseteq G' \text{ and } G'' \neq G' \\ L_{G''} \text{ for } G'' \in \mathbb{I}[G],\ G' \subseteq G'',\ G'' \neq G,\ G'' \neq G' \\ (L_{G''} - U_{G''}) \text{ for } G'' = G,\ G' \neq G \end{cases}$$

$$\bar{f}_G = \text{coefficient before } \gamma_\mu \text{ in } \tilde{f}_G$$

$$a_e = \sum_G \bar{f}_G$$

Details: S. Volkov, J. Exp. Theor. Phys. (2016), V. 122, N. 6, pp. 1008-1031

Example



$G_c = aa_1a_2b_1b_2c_1c_2c_3c_4$

$G_e = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3e_1e_2e_3$

$\mathbb{I}[G] = \{G_c, G_e, G\}$

Other UV-divergent subgraphs:

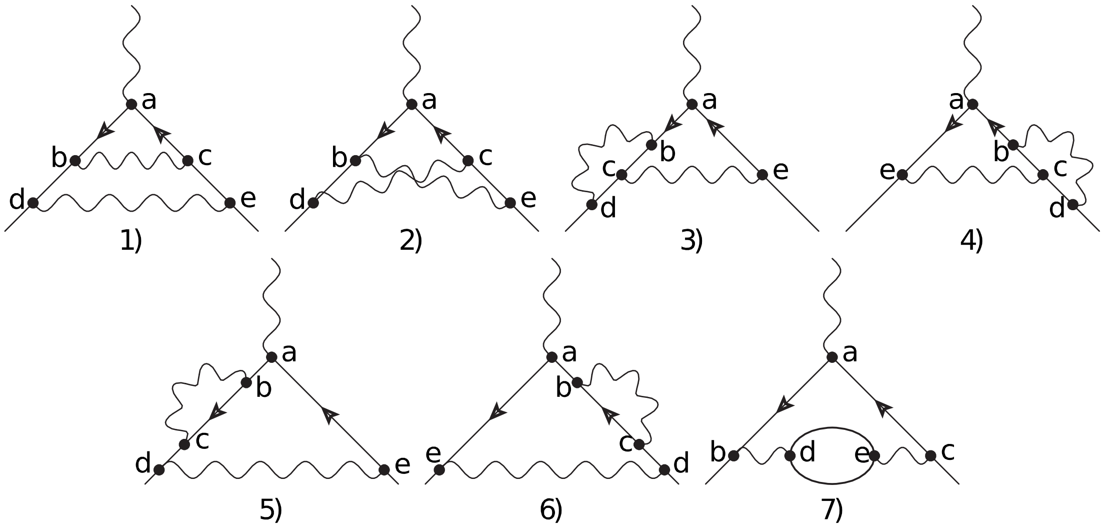
electron self-energy – a_1a_2, vertex-like – $c_1c_2c_3$, $c_1c_3c_4$,

photon self-energy – $c_1c_2c_3c_4$,

photon-photon scattering – $G_d = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3$

$$\tilde{f}_G = \left[A_G(1-U_{G_e})(1-U_{G_c}) - (L_G - U_G)A_{G_e}(1-U_{G_c}) - (L_G - U_G)(1-L_{G_e})A_{G_c}\right] \cdot$$
$$\cdot (1-U_{G_d})(1-U_{c_1c_2c_3c_4})(1-U_{c_1c_2c_3} - U_{c_1c_3c_4})(1-U_{a_1a_2}) f_G$$

Residual renormalization is not needed



$$B\Sigma(p) = a(m^2) + mb(m^2) +$$
$$+(\hat{p} - m)(b(m^2) + 2a'(m^2) + 2mb'(m^2)),$$
$$\Sigma(p) = a(p^2) + b(p^2)\hat{p}$$

#	Expression	On-shell renorm.	Difference
1	A_G-$A_G U_{abc}$-$(L_G$-$U_G)A_{abc}$	A_G-$A_G L_{abc}$	$(L_G$-$U_G)A_{abc}$-$A_G(L_{abc}$-$U_{abc})$
2	A_G	A_G	0
3	A_G-$A_G U_{bcd}$	A_G-$A_G L_{bcd}$	$A_G(U_{abc}$-$L_{abc})$
4	A_G-$A_G U_{bcd}$	A_G-$A_G L_{bcd}$	$A_G(U_{abc}$-$L_{abc})$
5	A_G-$A_G U_{bc}$	A_G-$A_G B_{bc}$	$A_G(U_{bc}$-$B_{bc})$
6	A_G-$A_G U_{bc}$	A_G-$A_G B_{bc}$	$A_G(U_{bc}$-$B_{bc})$
7	A_G-$A_G U_{de}$	A_G-$A_G U_{de}$	0

Importance sampling Monte Carlo

- Integral: $\int_\Omega f(x)dx$
- Probability density function: $g(x)$
- Approximation: $(1/N)\Sigma_{1\leq j\leq N}(f(x_j)/g(x_j))$
- Variance: $V(f,g)= \int_\Omega(f(x)^2/g(x))dx-(\int_\Omega f(x)dx)^2$
- Error estimation: $\sigma^2\approx V(f,g)/N$
- The goal is to minimize $V(f,g)$ by choosing $g(x)$.

NON-ADAPTIVE MONTE CARLO WORKS FINE FOR HIGH-ORDER CALCULATIONS IN QFT!!!

Importance sampling: example

- Integral: $\displaystyle\int_{0\le x_1,\dots,x_n\le 1} f(x_1,\dots,x_n)\,dx_1\dots dx_n$

$$f(x_1,\dots,x_n)=a_1\dots a_n x_1^{a_1-1}\dots x_n^{a_n-1}$$

- Density: $g(x_1,\dots,x_n)=b_1\dots b_n x_1^{b_1-1}\dots x_n^{b_n-1}$

- Variance: $V(f,g)=\dfrac{a_1^2\dots a_n^2}{b_1\dots b_n(2a_1-b_1)\dots(2a_n-b_n)}-1$

- All b_j are small => $V(f,g)$ is too big
- $b_j>2a_j$ for some j => $V(f,g)$ is infinite

Diagram-specific probability density functions

- Integral: $\int_{z_1,\ldots,z_M>0} f(z_1,\ldots,z_M)\delta(z_1+\ldots+z_M-1)dz$
- Hepp sectors: $z_{j_1} \geq z_{j_2} \geq \ldots \geq z_{j_M}$
- Density: $$C \cdot \frac{\prod_{l=2}^{M} (z_{j_l} / z_{j_{l-1}})^{Deg(\{j_l, j_{l+1},\ldots,j_M\})}}{z_1 \cdot z_2 \cdot \ldots \cdot z_M},$$

Deg is defined on subsets of {1,…,M}

(the idea of E.Speer, J. Math. Phys. 9, 1404 (1968))

- My ideas are:

1) how to calculate *Deg*(s) for each set s

(taking into account the infrared behavior etc.)

2) how to generate samples fastly

Obtaining *Deg*(s)

- Sector: $z_{j_1} \geq z_{j_2} \geq \ldots \geq z_{j_M}$
- Density: $C \cdot \dfrac{\prod_{l=2}^{M} (z_{j_l} / z_{j_{l-1}})^{Deg(\{j_l, j_{l+1}, \ldots, j_M\})}}{z_1 \cdot z_2 \cdot \ldots \cdot z_M}$,

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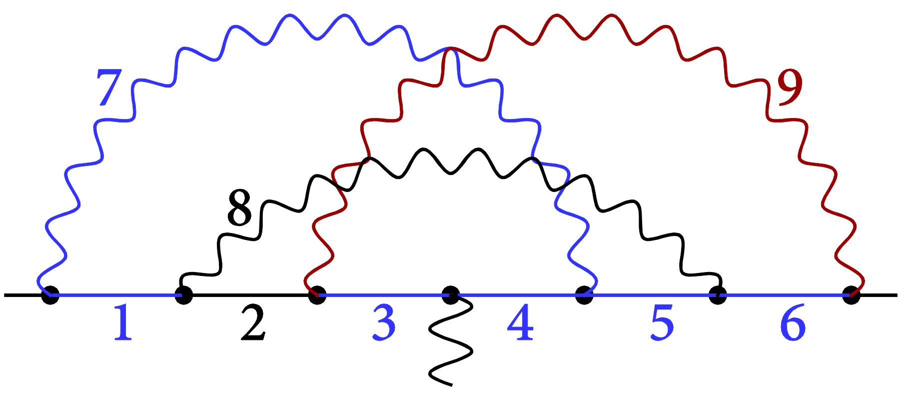
•The rules are constructed using ultraviolet degrees of divergence (with the sign '-') of I-closures of sets

(the full description taking into account divergent subdiagrams is in arXiv:1705.05800)

•IClos*(s)*=*s*U*s'*, where *s'* is the set of all photon lines for which the electron path connecting their ends is contained in *s*

Example: IClos({1,3,4,5,6,7})={1,3,4,5,6,7,9}

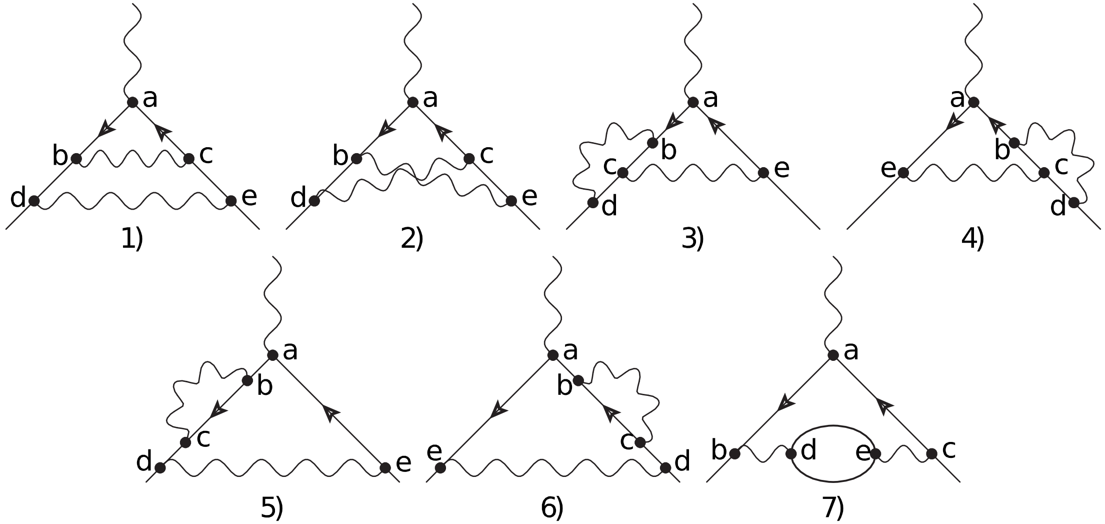


Realization and numerical results

- Monte Carlo integration on Intel-compatible CPUs, NVidia GPUs (Tesla K80, Tesla V100)
- 2 loops: all Feynman diagrams (with electron loops: old, 2015)
- 3 loops: all Feynman diagrams (with electron loops: old, 2015)
- 4 loops: diagrams without electron loops (GPU NVidia Tesla K80, Google Cloud)
- 5 loops: diagrams without electron loops (GPU NVidia Tesla V100, Govorun, JINR, Dubna)
- 6 loops: ladder diagram (NVidia Tesla K80, Google Cloud)

2 loops: all Feynman diagrams

NVidia Tesla K80, Google Cloud



#	My value	Analytical value (Petermann, 1957)
1	0.77747774(18)	0.77747802
2	-0.4676475(17)	-0.46764544
3,4	-0.0640193(19)	-0.564021–(1/2)log(λ²/m²)
5,6	-0.5899758(14)	-0.089978+(1/2)log(λ²/m²)
7	0.0156895(25)	0.0156874

2015: $A_1^{(4)} = -0.328513(87)$

2018: $A_1^{(4)}[\text{no lepton loops}] = -0.3441651(34)$

Analytical, 1957:

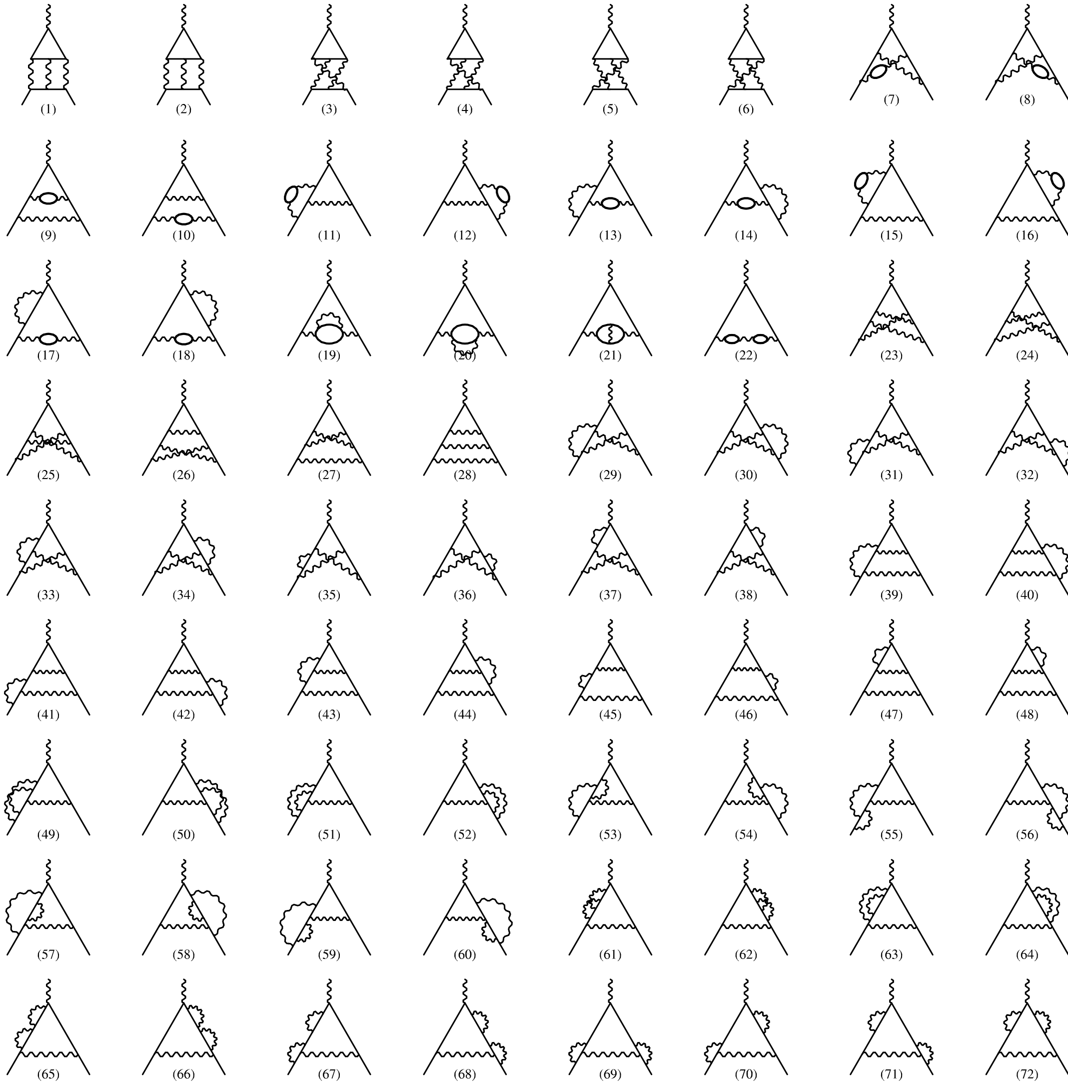
$$A_1^{(4)} = -0.328478966\ldots$$

$$A_1^{(4)}[\text{no lepton loops}] = -0.3441663\ldots$$

old: 2015, personal computer

3 loops: all Feynman diagrams $A_1^{(6)}[\text{no lepton loops}] = 0.90485(10)$

Analytical (1996):
0.904979



3-loop Feynman diagrams for electron's AMM. Plot courtesy of F.Jegerlehner

Comparison with known analytical values

old: 2015

#	My value	Analyt. val.	Reference
1-6	0.3708(14)	0.3710	[10]
7-10	0.04989(20)	0.05015	[4,5]
11-12,15-16	-0.08782(15)	-0.08798	[2,4]
13-14,17-18	-0.11230(17)	-0.11234	[3,4]
19-21	0.05288(13)	0.05287	[1]
22	0.002548(20)	0.002559	[1]
23-24	1.861914(17)	1.861908	[11]
25	-0.0267956(78)	-0.026799	[12]
26-27	-3.176700(22)	-3.176685	[8]
28	1.790285(19)	1.790278	[8]
29-30	-1.757945(15)	-1.757936	[12]
33-34,37-38	0.455517(26)	0.455452	[8,11]
31-32,35-36	1.541644(37)	1.541649	[7,9]
39-40	-0.334691(14)	-0.334695	[11]
41-48	-0.402749(46)	-0.402717	[6,7]
49-68	0.533289(54)	0.533355	[6-9,11,12]
69-72	0.421080(43)	0.421171	[6,7,9]

[1] J. Mignaco, E. Remiddi, IL Nuovo Cimento, V. LX A, N. 4, 519 (1969).
[2] R. Barbieri, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 5, N. 11, 769 (1972).
[3] D. Billi, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 4, N. 14, 657 (1972).
[4] R. Barbieri, E. Remiddi, Physics Letters, V. 49B, N. 5, 468 (1974).
[5] R. Barbieri, M. Caffo, E. Remiddi, Ref.TH.1802-CERN (1974).
[6] M. Levine, R. Roskies, Phys. Rev. D, V. 9, N. 2, 421 (1974).
[7] M. Levine, R. Perisho, R. Roskies, Phys. Rev. D, V. 13, N. 4, 997 (1976).
[8] R. Barbieri, M. Caffo, E. Remiddi et al., Nuclear Physics B 144, 329 (1978).
[9] M. Levine, E. Remiddi, R. Roskies, Phys. Rev. D, V. 20, N. 8, 2068 (1979).
[10] S. Laporta, E. Remiddi, Physics Letters B 265, 182 (1991).
[11] S. Laporta, Physics Letters B 343, 421 (1995).
[12] S. Laporta, E. Remiddi, Physics Letters B 379, 283 (1996).

4 loops: diagrams without electron loops

My result: -2.181(10)   1 week on 1 GPU (from NVidia Tesla K80, Google Cloud)

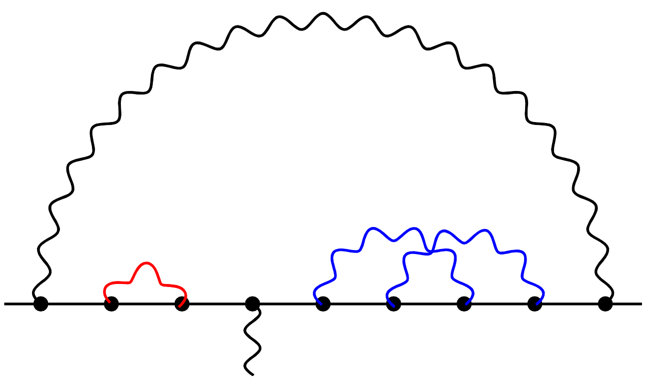
Laporta, 2017: -2.1768660277…

- 269 Feynman diagrams
- 78 classes of diagrams for comparison with the direct subtraction on the mass shell
- 6 gauge-invariant classes (k,m,n)

(k,m,n):

m and n photon lines to the right and to the left from the external photon (or vice versa),
k photon lines with ends on different sides

Example of a diagram from (1,2,1):



Class	Value	Laporta, 2017
(1,3,0)	-1.9710(44)	-1.97107…
(2,2,0)	-0.1415(56)	-0.14248…
(1,2,1)	-0.6220(46)	-0.62192…
(3,1,0)	-1.0424(44)	-1.04054…
(2,1,1)	1.0842(37)	1.08669…
(4,0,0)	0.5120(17)	0.51246…

5 loops: diagrams without electron loops

T. Aoyama, T. Kinoshita, M. Nio, 2017 (90% confidence): 7.606(192)

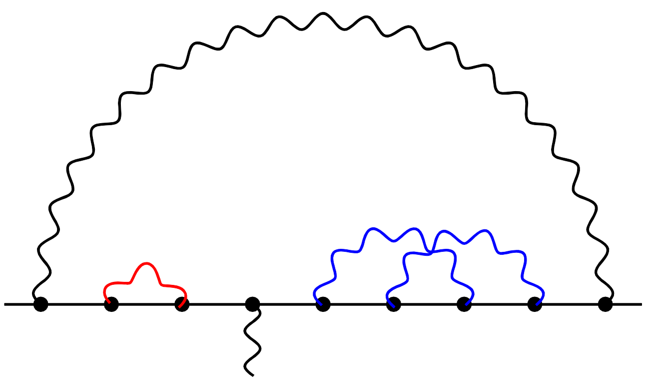
My result (1σ): **6.641(227)** 8656 GPU-hours, NVidia Tesla V100, supercomputer "Govorun" (JINR, Dubna)

- 3213 Feynman diagrams
- 807 classes of diagrams for comparison with the direct subtraction on the mass shell
- 9 gauge-invariant classes (k,m,n)
- 500 GB of the integrands code (compiled)
- $6.5\cdot 10^{13}$ Monte Carlo samples

(k,m,n):

m and n photon lines to the right and to the left from the external photon (or vice versa), k photon lines with ends on different sides

Example of a diagram from (1,2,1):



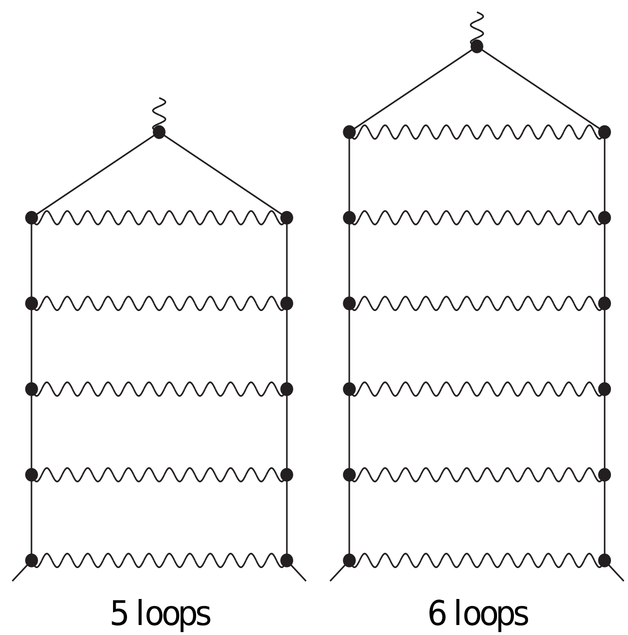
Class	Value=Σx_j	N_{diag}	$\Sigma\|x_j\|$	$\max\|x_j\|$
(1,4,0)	6.180(84)	706	1219.7	11.8
(2,3,0)	-0.81(11)	706	3076.8	46.2
(1,3,1)	0.747(87)	148	3170.3	67.5
(3,2,0)	-0.414(87)	558	2593.5	54.9
(2,2,1)	-2.100(92)	370	3318.0	85.0
(4,1,0)	-1.056(52)	336	1199.3	56.7
(1,2,2)	0.361(50)	55	1338.4	68.7
(3,1,1)	2.642(61)	261	1437.2	63.5
(5,0,0)	1.091(15)	73	137.0	19.3

Ladder diagrams: 5 and 6 loops

(NVidia Tesla K80 (1 GPU), Google Cloud)

loops	My value	Analytical value	$N_{samples}$	time
5	11.6530(58)	11.6592…	$29 \cdot 10^9$	5 hours
6	34.31(20)	34.367	10^{10}	8 hours

All analytical values are from M. Caffo, S. Turrini, E.Remiddi, Nuclear Physics B141 (1978) 302-310.



Thank you for your attention!

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ЖЭТФ, т. 149, вып. 6, стр. 1164-1191 (2016)

J. Exp. Theor. Phys. 122, 1008 (2016)

arXiv:1507.06435 (short version)

subtraction procedure

Phys. Rev. D 96, 096018 (2017)

arXiv:1705.05800

Monte Carlo integration method

Phys. Rev. D 98, 076018 (2018)

arXiv:1807:05281

realization on GPU, 4-loop results for gauge-invariant classes, …